Weighted Digraphs and its Spectrum

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Abstract-

This paper explores the concept of Weighted Digraphs associated with cyclic groups Z_n. In these digraphs, each arc is assigned a weight based on modular arithmetic, specifically the smallest integer r such that $y \equiv r \cdot x \pmod n$, where x and y are vertices of the digraph. The study discusses various properties of these weighted digraphs, including the relationship between the weight of arcs and the order of elements in Z_n, the behavior of generators, and the uniqueness of arc weights. The adjacency matrix and degree matrix are defined, and the Laplacian matrix is derived as the difference between these two matrices. Several examples are presented for Z_2, Z_3, Z_4, Z_5, and Z_6, showcasing their adjacency matrices, degree matrices, Laplacian matrices, characteristic polynomials, and eigenvalues. The results provide insight into the structure and properties of weighted digraphs on cyclic groups and demonstrate the use of software tools such as MATLAB for matrix computations.

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1 Introduction

Manuel and Thomas [9] defined the concept of a new digraph associated with the finite cyclic group Zn. The digraph $Zn-digraph$ or simply a simple digraph with vertex set Zn and two distinct vertices $u, v \in Zn$ are joined by a directed edge or an arc $\overrightarrow{uv}$ from u to v if and only if there exists a non-negative integer r such that $v \equiv ru(\text{mod}\, n)$. It is proved that for any n, the in-degree and out-degree of the vertex 0 in Zn, are $n-1$ and 0 respectively, and if a is a generator of the group Zn, then $id(a) = \phi(n) - 1$ and $od(a) = n-1$ in Zn, where $\phi(n)$ is the number of generators of Zn and if $0 \neq b \in V$ and b is not a generator of Zn, then $id(b) = \phi(n) + \phi(h_1) + \cdots + \phi(h_r) - 1$, where h_i is the order of the cyclic subgroup generated by a_i. Here a_i is not a generator of $Zn, b \in < a_i >$ and $< a_i > \neq < a_j >$, for all $i \neq j$.
In this paper, we define and study the Weighted Z_n Digraph, its spectrum, and the Laplacian spectrum.

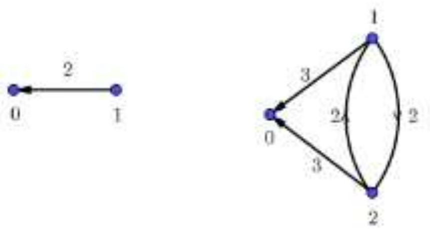
2 Weighted Digraph

Definition 2.1. *Weighted Z_n digraph* is a Z_n digraph in which we give a particular weight to each of the arcs. The weight given to the arc $\overrightarrow{xy}$ is the smallest (in case it is not unique) r such that

$$y \equiv rx(\text{mod} n),\ 1 \leq r \leq n.$$

Figure 1 gives the examples of weighted digraphs associated with Z_2 and Z_3.

- In Z_2, $0 \equiv (2)(1) \pmod 2$ and
- In Z_3, $0 \equiv (3)(1) \pmod 3$, $0 \equiv (3)(2) \pmod 3$, $1 \equiv (2)(2) \pmod 3$ and $2 \equiv (2)(1) \pmod 3$



***Observations* 2.2**. The weight of the arc $\overrightarrow{x0}$ in weighted $\overrightarrow{Z_n}$ digraph is the same as the order of x in Z_n. That is, $\frac{n}{(x, n)}$. So the weight of the arcs $\overrightarrow{a_0}$ in weighted $\overrightarrow{Z_n}$ digraph is n if a is a generator of Z_n.

***Observations* 2.3**. The weight of the arc $\overrightarrow{1x}$ in weighted $\overrightarrow{Z_n}$ Digraph

is

$$\begin{cases} x, & \text{if } x \neq 0 \\ n, & \text{if } x \neq 0 \end{cases}$$

Observations 2.3. Let a be a generator of Z_n. Then the weight of the arc $\overrightarrow{ab}$ in the weighted $\overrightarrow{Z_n}$ digraph is $ba^{-1} (\bmod n)$.

***Observations* 2.4**. Let $\overrightarrow{ab}$ be an arc in $\overrightarrow{Z_n}$ digraph and suppose a is not a generator of Z_n, then $\gcd(a, n) \neq 1$, let it be d. Since there exist an arc $\overrightarrow{ab}$, there is an r such that $b \equiv ar (\bmod n)$. But it may not be unique. For example, in Z_{12} for the arc $\overrightarrow{24}$,

$4 \equiv (2)(2)(\bmod 12)$ and $4 \equiv (8)(2)(\bmod 12)$. But we take 2 as the weight, $\min\{2, 8\}$.

Observations 2.5. Since $x, y \in Z_n$, it is not possible that $y \equiv 1.x(\bmod n)$. So 1 is not a weight of any arc in $\overrightarrow{Z_n}$.

***Theorem* 2.5**. Let $0 \neq a \in Z_p$, then there exists $p-1$ arcs from a to b, $\forall b \in Z_p - \{a\}$. Then the weights r of the arcs $\overrightarrow{ab}$ such that $b \equiv ar (\bmod p)$ are exactly one from the set $\{2, 3, \ldots, p\}$ of cardinality $p-1$ in some order.

Proof. Since there exist an arc $\overrightarrow{ab}$, $\exists r$ such that $b \equiv ar(\bmod p)$.

Since $g.c.d(a, p) = 1$, $r \equiv ba^{-1}(\bmod n)$. We have every integer is congruent modulo p to exactly one of the values from the set $\{1, 2, \ldots, p\}$.

But by the above observation, is not a weight of any arc in $\overrightarrow{Z_n}$. Hence weights are exactly from the set $\{2, 3, \ldots, p\}$.

Suppose these weights are not different. Suppose the arcs $\overrightarrow{ab_1}$ and $\overrightarrow{ab_2}$ have same weight, so $b_1 \equiv ar(\bmod p)$ and $b_2 \equiv ar(\bmod p)$, then $b_1 \equiv b_2(\bmod p)$, which is not possible

***Observations* 2.6.** In weighted $\overrightarrow{Z_p}$ digraph each arc $\overrightarrow{ab}$ has a unique weight. Suppose the arc $\overrightarrow{ab}$ has two weights r_1 and r_2, then $b \equiv ar_1(\bmod p)$ and $b \equiv ar_2(\bmod p)$. Since $g.c.d(a, p) = 1$, $r_1 \equiv r_2(\bmod p)$, hence $r_1 = r_2$. So, the weights are unique.

***Theorem* 2.7.** Let $a \in Z_n$, then $n-a$ be the inverse of a in Z_n. Then there exist arcs $\overrightarrow{a(n-a)}$ and $\overrightarrow{(n-a)a}$. Moreover, the weights of these arcs are the same.

Proof. Since a and $n-a$ are inverses of each other they have the same order. Therefore $< a >$ and $< n-a >$ are the same. Since $x \equiv (n-1)(n-x)(\bmod n)$ and $n-x \equiv (n-1)x(\bmod n)$, There exist arcs $\overrightarrow{a(n-a)}$ and $\overrightarrow{(n-a)a}$.

Since there exists an arc $\overrightarrow{a(n-a)}$ there exist r_1 such that

$$(n-a) \equiv ar_1(\bmod n)$$

$$n - a - r_1 a \equiv 0(\text{mod}\, n)$$

$$a + r_1 a \equiv 0(\text{mod}\, n)$$

$$-a \equiv r_1 a(\text{mod}\, n)$$

If $g.c.d(a, n) = 1$, then

$$-1 \equiv r_1(\text{mod}\, n)$$
$$r_1 \equiv -1 \equiv n - 1(\text{mod}\, n)$$

Since there exists an arc $a\,(n - a)$, there exist r_1 such that
$$(n - a) \equiv a r_1(\text{mod}\, n)$$

Then as above
$$r_2 \equiv -1 \equiv n - 1(\text{mod}\, n)$$

If $g.c.d(a, n) = d \neq 1$. Then,

$$r_1 a \equiv -x(\text{mod}\, n)$$

$$r_1(a/d) \equiv -(x/d)\ (\text{mod}(n/d))$$

$$r_1 \equiv -1(\text{mod}(n/d))$$

$$r_1 \equiv (n/d) - 1(\text{mod}(n/d))$$

Similarly

$$r_2 \equiv (n/d) - 1(\text{mod}(n/d))$$

So, the weight is the same for the arcs $\overrightarrow{a(n-a)}$ and $\overrightarrow{(n-a)a}$.

3 The adjacency matrix of a weighted digraph

The *adjacency matrix* of a weighted digraph G with weight w_{ij} of an arc $\overrightarrow{ij}$ is $A(G) = [a_{ij}]$, where,

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } \overrightarrow{ij} \in E(G), \\ 0, & \text{otherwise.} \end{cases}$$

The degree matrix $D = [d_{ij}]$ is the matrix with

$$d_{ij} = \begin{cases} d(i), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Here $d(i)$ is the sum of weights of i^{th} row and i^{th} column. That is,

$$d_{ii} = d(i) = \sum_{j=1}^{n} w_{ij} + \sum_{k=1}^{n} w_{ki}.$$

For example, the adjacency matrix of weighted $\overrightarrow{Z_2}$ digraph is

$$A(\overrightarrow{Z_2}) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$$

and Degree matrix of weighted $\overrightarrow{Z_2}$ digraph is

$$D(\overrightarrow{Z_2}) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

The adjacency matrix of weighted $\overrightarrow{Z_3}$ digraph is

$$A(\overrightarrow{Z_3}) = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

and Degree matrix of weighted $\overrightarrow{Z_3}$ digraph is

$$D(\overrightarrow{Z}_3)=\begin{bmatrix}6&0&0\\0&7&0\\0&0&7\end{bmatrix}.$$

The adjacency matrix of weighted $\overrightarrow{Z}_4$ digraph is

$$A(\overrightarrow{Z}_4)=\begin{bmatrix}0&0&0&0\\4&0&2&3\\2&0&0&0\\4&3&2&0\end{bmatrix}$$

and Degree matrix of weighted $\overrightarrow{Z}_4$ digraph is

$$D(\overrightarrow{Z}_4)=\begin{bmatrix}10&0&0&0\\0&12&0&0\\0&0&6&0\\0&0&0&12\end{bmatrix}.$$

The adjacency matrix of weighted $\overrightarrow{Z}_5$ digraph is

$$A(\overrightarrow{Z}_5)=\begin{bmatrix}0&0&0&0&0\\5&0&2&3&4\\5&3&0&4&2\\5&2&4&0&3\\5&4&3&2&0\end{bmatrix}.$$

The adjacency matrix of weighted $\overrightarrow{Z}_6$ digraph is

$$A(\overrightarrow{Z}_6)=\begin{bmatrix}0&0&0&0&0&0\\6&0&2&3&4&5\\3&0&0&0&2&0\\2&0&0&0&0&0\\3&0&2&0&0&0\\6&5&4&3&2&0\end{bmatrix}.$$

The adjacency matrix of weighted $\overrightarrow{Z}_7$ digraph is

$$A(\overrightarrow{Z}_7)=\begin{bmatrix}0&0&0&0&0&0&0\\7&0&2&3&4&5&6\\7&4&0&5&2&6&3\\7&5&3&0&6&4&2\\7&2&4&6&0&3&5\\7&3&6&2&5&0&4\\7&6&5&4&3&2&0\end{bmatrix}.$$

Using the software *MATLAB*, we calculated the characteristic polynomial and eigenvalues of some of these adjacency matrices as follows:

- For $A(Z_2)$, the characteristic polynomial is x^2 and the eigen values are $0, 0$.
- For $A(Z_3)$, the characteristic polynomial is $x^3 - 4x$ and the eigenvalues are $0, 2, -2$.
- For $A(Z_4)$, the characteristic polynomial is $x^4 - 9x^2$ and the eigenvalues are $0, 0, 3, -3$.
- For $A(Z_5)$, the characteristic polynomial is $x^5 - 56x^3 - 208x^2 - 153x$ and the eigenvalues are $0, 9, -1, -4 + i, -4 - i$
- For $A(Z_6)$, the characteristic polynomial is $x^6 - 29x^4 + 100x^2$ and the eigenvalues are $0, 0, 2, -2, 5, -5$.

Now the Laplacian matrix is given by

$$L(Z_n) = D(Z_n) - A(Z_n).$$

$$L(\vec{Z}_2) = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}.$$

$$L(\vec{Z}_3) = \begin{bmatrix} 6 & 0 & 0 \\ -3 & 7 & -2 \\ -3 & -2 & 7 \end{bmatrix}.$$

$$L(\vec{Z}_4) = \begin{bmatrix} 10 & 0 & 0 & 0 \\ -4 & 12 & -2 & -3 \\ -2 & 0 & 6 & 0 \\ -4 & -3 & -2 & 12 \end{bmatrix}.$$

$$L(\vec{Z}_5) = \begin{bmatrix} 20 & 0 & 0 & 0 & 0 \\ -5 & 23 & -2 & -3 & -4 \\ -5 & -3 & 23 & -4 & -2 \\ -5 & -2 & -4 & 23 & -3 \\ -5 & -4 & -3 & -2 & 23 \end{bmatrix}.$$

Using the software *MATLAB*, we calculated the characteristic polynomial and eigenvalues of some of these Laplacian matrices as follows:

- For $L(Z_2)$, the characteristic polynomial is $x^2 - 4x + 4$ and the eigen values are 2, 2.
- For $L(Z_3)$, the characteristic polynomial is $x^3 - 20x^2 + 129x - 270$ and the eigenvalues are 9, 5, 6.
- For $L(Z_4)$, the characteristic polynomial is $x^4 - 40x^3 + 579x^2 - 3600x + 8100$ and the eigenvalues are 15, 9, 6, 10.
- For $L(Z_5)$, the eigenvalues are $14, 24, 20, 27 + i, 27 - i$

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